SIMULATIONS OF PROBABILITIES FOR QUANTUM COMPUTING

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SUMMARY

It has been demonstrated that classical probabilities, and in particular, probabilistic Turing machine, can be simulated by combining chaos and non-Lipschitz dynamics, without utilization of any man-made devices (such as random number generators). Self-organizing properties of systems coupling simulated and calculated probabilities and their link to quantum computations are discussed.

Classical dynamics is fully deterministic if initial conditions are known exactly. Otherwise in some non-linear systems, small initial errors may grow exponentially so that the system behavior attains stochastic-like features, and such a behavior is called chaotic. The discovery of chaos contributed in better understanding of irreversibility in dynamics, of evolution in nature, and in interpretation and modelling of complex phenomena in physics and biology. However, there is a class of phenomena which cannot be represented by chaos directly. This class includes so called discrete events dynamics where randomness appears as point events, i.e., there. is a sequence of random occurences at fixed or random times, but there is no additional component of uncertainty between these times. The simplest example of such a phenomenon is a heartbeat dynamics which, in the first approximation, can be modelled by a sequence of pulses of equal heights and durations, but the durations of the pauses between these pulses are randomly distributed. Most processes of this type are associated with intellectual activities such as optimal behavior, decision making process, games, etc. In general, discrete events dynamics is characterized by a well-defined probabilistic structure of a piecewise-deterministic Markov chains, and it can be represented by probabilistic Turing machine. On the contrary, a probabilistic structure of chaos, and even the appearance of chaos at all, cannot be predicted based only upon the underlying model without actual numerical runs. (The last statement can be linked to the Richardson's [1] proof that the theory of elementary functions in classical analysis is undecidable). But is there a "missing link" between chaos and discrete events dynamics? And if it is, can this link be simulated based only upon physical laws without exploiting any man-made devices such as random number generators? A positive answer to this question would make a fundamental contribution to the reductionists view on intrinsic unity of science that all natural phenomena are reducable to physical laws. However, in addition to this philosophical aspect, there is a computational advantage in exploiting simulated probabilities instead of calculated ones in the probabilistic Turing machine: as shown by R. Feynman [2], the exponential complexity of algorithms in terms of calculated probabilities can be reduced to polynomial complexity in terms of simulated probabilities.

In this paper we demonstrate that the missing link between chaos and a discrete event process can be represented by non-Lipschitz dynamics. [3-5]

in order to illustrate. the basic concepts of non-Lipschitz dynamics, consider a rectilinear motion of a particle of unit mass driven by a non-Lipschitz force:

$$\dot{\upsilon} = \nu\upsilon^{1/3}\sin \omega t,\ \nu = const, \qquad [\nu] = -\frac{m^{2/3}}{\sec^{5/3}} \tag{1}$$

$$\dot{x} = \upsilon \tag{2}$$

where v and x are the particle velocity and position, respectively.
Subject to the zero initial condition

$$v = 0 \qquad \text{at} \qquad t = 0 \tag{3}$$

equation (1) has a singular solution

$$v = 0 \tag{4}$$

and a regular solution

$$\upsilon = \pm\left(\frac{4\nu}{3\omega}\sin^2\frac{\omega}{2}t\right)^{3/2} \tag{5}$$

These two solutions coexist at t = 0, and this is possible because at this point the Lipschitz condition fails:

$$\left|\frac{\partial \dot{\upsilon}}{\partial \upsilon}\right|_{t\to 0} = \frac{1}{3}\nu\upsilon^{-2/3}\sin\omega t\Big|_{t\to 0} \to \infty \tag{6}$$

Since

$$\frac{\partial \dot{\upsilon}}{\partial \upsilon} > 0 \qquad \text{at } |\upsilon| \neq 0, \qquad t > 0 \tag{7}$$

the singular solution (4) is unstable, and the particle departs from rest following the solution (5). This solution has two (positive and negative) branches [since the power in (5) includes the square root], and each branch can be chosen with the probability p and (1 -p) respectively. It should be noticed that as a result of (5), the motion of the particle can be initiated by infinitesimal disturbances (such motion never can occur when the Lipschitz condition holds: an infinitesimal initial disturbance cannot become finite in finite time).

Strictly speaking, the solution (5) is valid only in the time interval

$$0 \leq t \leq \frac{2\pi}{\omega} \tag{8}$$

and at t $= 2\pi / \omega$ it coincides with the singular solution (4)

For $t > 2\pi / \omega$ equation (4) becomes unstable, and the motion repeats itself to the accuracy of the sign in equation (5).

Hence, the particle velocity υ performs oscillations with respect to its zero value in such a way that the positive and negative branches of the solution (5) alternate randomly after each period equal to $2\pi / \omega$.

Turning to equation (2), one obtains the distance between two adjacent equilibrium position of the particle:

$$x_i - x_{i-1} = \pm \int_0^{2\pi/\omega} \left(\frac{4\upsilon}{3\omega} \sin \frac{\omega}{2} t \right)^{3/2} dt = 64(3\omega)^{-5/2} v^{3/2} = \pm h \tag{9}$$

Thus, the equilibrium positions of the particle, are

$$x_0 = 0, \qquad x_1 = \pm h, \qquad x_2 = \pm h \pm h \ldots \tag{10}$$

while the positive and negative signs randomly alternate with probabilities p and (1-p), respectively.

Obviously, the particle performs an unrestricted random walk: after each time period

$$\tau = \frac{2\pi}{\omega} \tag{11}$$

it changes its value on $\pm h$ [see equation (10)].

The probability density f(x,t) is governed by the following difference equation:

$$f(x, t + \tau) = pf(x - h, t) + (1 - p) f(x + h, t) \tag{12}$$

which represents a discrete version of the Fokker-Planck equation, while

$$\int_{-\infty}^{\infty} f(x, t)\, dx = 1 \tag{13}$$

Several comments to the model (1) and its solution have to be made.

Firstly, the "viscous" force

$$F = -v\upsilon^{1/3} \tag{14}$$

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includes static friction (see Eq. 6) which actually causes failure of the Lipschitz condition. These type of forces are well-known in theory of visco-plasticity [6]. It should be noticed that the power 1/3 can be replaced by any power Of the type:

$$k = \frac{2n-1}{2n+i}, \text{ n } = 1,2,\ldots \text{ etc} \tag{15}$$

with the same final result (1 2). in particular, by selecting large n, one can make k close to 1, so that the force (13) will be almost identical to its classical counterpart

$$F_c = -\nu\upsilon \tag{16}$$

everywhere excluding a small neighborhood of the equilibrium point v = O, while at this point

$$\frac{dF}{d\upsilon} \to \infty, \qquad \text{but } \left|\frac{\partial F_c}{\partial \upsilon}\right| \to 0 \qquad \text{at} \qquad \upsilon \to 0 \tag{17}$$

Secondly, without the failure of the Lipschitz condition (6), the solution to Eq. (1) could not approach its equilibrium υ = O in finite time, and therefore, the paradigm leading to random walk (12) would not be possible.

Finally, we have to discuss the infinitesimal disturbances mentioned in connection with the instability of the solutions (5) at υ = O. Actually the original equation should be written in the form:

$$\dot{\upsilon} = \nu\upsilon^{1/3} \sin\omega t + \varepsilon(t), \ \varepsilon \to 0 \tag{18}$$

where $\varepsilon(t)$ represents a time series sampled from an underlying stochastic process representing infinitesimal disturbances. It should be emphasized that this process is not driving the solution of Eq. (1 8): it only triggers the mechanism of instability which controls the energy supply via the harmonic oscillations $\sin\omega t$. As follows from Eq. (1 8), the. function $\varepsilon(t)$ can be ignored when $\dot{\upsilon} = 0$ or when $\dot{\upsilon} \neq$ O, but the equation is stable, i.e. $\upsilon = \pi\omega, 2\pi\omega, \ldots$ etc. However, it becomes significant during the instants of instability when $\dot{\upsilon}$ = O at $t = 0, \pi/2\omega$ etc. Indeed, at these instants, the solution to Eq. (1) has a choice to be positive or negative if ε = O, (see eq. (5)). However, with $\varepsilon \neq$ O,

$$sign\, x = sign\, \varepsilon \ \text{ at } \ t = 0, \ \pi/2\omega, \ldots \text{etc} \tag{19}$$

i.e., the sign of ε at the critical instances of time (19) uniquely defines the evolution of the dynamical system (1 8). ~'bus, the dynamical system (18) transforms a stochastic process (via its sample $\varepsilon(t)$) into a binary time series which, in turn, simulates a random-walk (1 8). Actually the solution to eq. (18) represents a statistical signature of the stochastic process ε .

Within the framework of dynamical formalism, the time series $\varepsilon(t)$ can be generated by a fully deterministic (but chaotic) dynamical system. The simplest of such system is the logistic map which plays a central role in population dynamics, chemical kinetics anti many other fields. in its chaotic domain

$$y_{n+1} = 4y_n(1-y_n), \; y_0 = 0.2 \qquad \textbf{(20)}$$

the power spectrum for the solution is indistinguishable from a white noise. However, for the better match with Eq. (1 8), we will start with a continuous version of (20) represented by the following time-delay equation.

$$y(t + \tau) = 4y(t)[1 - y(t)], \; \tau = \; ;:- \qquad (21)$$

$$y(t^{*}) = 0.2, \; -\frac{\pi}{4\omega} \langle t^{*} \langle \frac{\pi}{4\omega} \qquad (22)$$

The solution to Eq.(21) at t=0, $\pi / 2\omega$,... etc, coincides with the solution to Eq. (20), but due to the specially selected initial condition (22), the solution to Eq.(20) changes its values at $t = \frac{\pi}{4\omega}, - \frac{\tau}{4@}, \ldots$ etc, so that at the points t=0, π /2[0,... , the sign of this solution is well-clefinul.

Now assume that

$$\varepsilon(t) = \varepsilon_0(y(t) - 0{,}51), \; \varepsilon_0 \;((\; 1. \qquad (23)$$

The substraction from y(t) its mean value provides the condition

$$p = 1 - p = \frac{1}{2} \qquad (24)$$

Indeed, for the first hundred points in (23),

$$Sign\;\varepsilon = \begin{array}{l} -++-+++--+--+-------++++--+ \\ --++-+-+--++-----+-+----++-+ \\ +-++-+++---++++++--++++++--+ \\ +-----++----+-+----+--- \end{array} \qquad (25)$$

has equal number of positive and negative values which are practically not correlated. Therefore, the statistical signature of the chaotic time series (23) is expressed by the solution to Eqs (12), (13) at $p = \frac{1}{2}$ with the initial conditions

$$f(0,0) = 1, f(x,0) = O \text{ if } x \neq O \qquad (26)$$

which is a symmetric unrestricted random walk:

$$f(x,t) = C_n^m 2^{-n}; \quad m = -\frac{1}{2}(n \text{ -t } x); \; n = \text{integer} \frac{2\omega t}{(\;\pi\;)} \qquad (27)$$

Here the binomial coefficient should be interpreted as 0 whenever m is not an integer in the interval $[0, n]$ and n is the total number of steps.

The connection between the solution (26) and the solutions to the system (18), (21), (2) should be understood as follows. Suppose we solve the system (1 8), (21), (2) subject to the intial condition (2'2) with $v = 0$ and $x =$ O $at\ t =$ O. Since Eq. (21) is supersensitive to inevitable errors in (22) the solution will form an ensemble of chaotic time series, and for any fixed instant of time this ensemble will have the corresponding probability density distribution which coincides with (26). In other words, the probabilities described by Eq. (1 2), are simulated by the dynamical system (1 8), (21) and (2) without an explicit source of stochasticity (while the "hidden" source of stochasticity is in finite precision of the initial condition (22)).

Combining several dynamical systems of the type (1 8), (21), (2) and applying an appropriate change of variables, one can simulate a' probabilistic Turing machine which transfers one state to another with a prescribed transitional probabilities, [3]. Non-Markovian properties of such a machine can be incorporated by introducing time-delay terns in Eq. (2).

$$\dot{x} = v(t) + \alpha_1 v(t - \tau_0) + \alpha_2 v(t - 2\tau_0) + \ldots \qquad (28)$$

However, there is a more interesting way to enhance the dynamical complexity of the system (1 8), 21), (2). lmiee.et, let us turn to Eq. (23) and introduce a feedback from Eq. (2) to Eq. (18) as following:

$$\varepsilon = \varepsilon_0(\tilde{y} - x), \varepsilon_0 << 1, \tilde{y} = y - 0.51 \qquad (29)$$

Then the number of negative (positive) signs in the string (25) will prevail if x) O (x (O) since the effective zero-crossing line moves down (up) away from the middle. ~'bus, when $(x = 0)$ at $t = 0$, the system starts with an unrestricted random walk as described above, and $|x|$ grows. However, this growth changes signs in Eq. (23) such that $\dot{x}$ (O if x) 0, and $\dot{x}$) O if x (O. As a result of that

$$x_{max} \le y_{max}, \quad x_{min} \ge y_{min} \qquad (30)$$

where x_{max} and y_{min} are the largest and the smallest values in the time series y (t), respectively. Hence, the dynamical system (1 8), (23), (2) simulates a restricted random walk with the boundaries (30) implemented by the dynamical feedback (29), while the probability

$$p(sign\varepsilon\rangle 0) \quad \begin{cases} 0 \text{ if } x = y_{max} \\ 1 \text{ if } x = y_{min} \end{cases} \qquad (31)$$

For the sake of qualitative discussion, assume that p change linearly between $x = y_{min}$ and $x = y_{max}$, i.e.,

$$p = \begin{cases} 0 & \text{if } x > y_{max} \\ -\frac{y_{max} - x}{y_{max} - y_{min}} & \text{if } y_{min} \leq x \leq y_{max} \\ 1 & \text{if } x < y_{min} \end{cases} \tag{32}$$

(the actual function p(x) depends upon statistical properties of the underlying chaotic time series y(t)),

Then the simulated restricted random walk as a solut ion to Eqs. (12) and (32.) Let us modify the feedback (29) as

$$\varepsilon = \varepsilon_0\left[\tilde{y} - \left(x^2 - x\right)\right] \tag{33}$$

Now when x=0 at t=O, the system is unstable since

$$sgn\, x = sgn\, \dot{x}, \quad -\infty \langle\, x \,\langle\, \frac{1}{2}, \tag{34}$$

and the process is divided into two branches. The negative branch (with the probability 1/2) represent an unrestricted random walk ($x \to \infty$), while the positive branch (with the same probability ½) is eventually trapped within the basin of the attractor x = 1 since

$$sgn\, x = -sgn\, \dot{x}, \quad \frac{1}{2} \,\langle\, x \,\langle\, \infty \tag{35}$$

simulating a restricted random walk as those described above with the only difference that its center is shifted from x = O to x = 1.

As a next step in complexity, introduce the information H associated with the random walk process described by Eqs. (1 2), (1 3):

$$H = -\int_{-\infty} f \log_2 f \, dx \tag{36}$$

and modify the feedback (29) as following:

$$\varepsilon = \varepsilon_0\left[\tilde{y} - x(]+ 2f\log_2 f)\right] \tag{37}$$

where $-f\log_2 f$ is the information per unit step of x.

Following the same line of agrumentation as those performed for the feedback (29), one concludes that the feedback (38) becomes active only if the process is out of the domain of the maximum information, and therefore, it is always attracted to this domain.

Since Eq. (31) is still valid, we will apply the approximation similiar to (32):

$$p = \begin{cases} 0 \text{ if } x(1+H) \geq y_{max} \\ \dfrac{y_{max} - x(1+H)}{y_{max} - y_{min}} & \text{if } y_{min} \leq x(1+H) \leq y_{max} \\ 1 \text{ if } x(1+H) \leq y_{min} \end{cases} \quad (38)$$

in order to continue our qualitative analysis. It should be noticed that now p depends not only on x, but also on f, and that makes Eq. (12) nonlinear. in addition to that, the system (18), *(2)* and *(37),* which is simulating probabilities, is coupled with the system (12), (13) and (38) describing the evolution of calculated probabilities. Actually due to this coupling, the entire dynamical system attains such a self-organizing property as to maximize the information generated by the random walk.

The self-org,anizing properties of the system (18), (2), (37), (12), (13) and (38) mentioned above have a very interesting computational interpretation: they provide a mutual influence between different branches of probabilistic scenarios. Such an influence or interference, is exploited in hypothetical quantum computer [7] as a more powerful tool in a complexity theoretic sense, than classical probabilistic computations. However, in quantum computer, the interference is restricted to a unitary matrix transformation of probabilities (which is the only one allowed by quantum mechanics laws), while in the classical system (18), *(2), (37)* there is no such restriction; by choosing an appropriate probabilistic term in the feedback (37), we can provide an optimal interference. Unfortunately, the prim paid for such a property is the necessity to exploit the calculated probabilities (12), (13) and (38), which, in many cases is a significant disadvantage.

in conclusion, we will briefly summarize the results, Firstly, it has been demonstrated that a differential dynamical system which couples a simple generator of chaos and a non-Lipschitz dynamical device can simulate stochastic processes of prescribed complexity without exploiting man-matte devices (such as step-functions, random number generators, etc.) i.e., based only upon physical laws, The last property is important for physical interpretation of non-deterministic biological processes (heartbeats, breathing periods, electroencephalograms).

Secondly, a new level of dynamical complexity was introduced by coupling the non-Lipschitz system (1), (2) and the associated probability equation (12) representing a discrete version of the Folker-Planck equation. This sytem attains emerging self-organization properties which can be utilized for synthesis of intelligent systems simulating optimal behavior, decision making processes, games, etc.

Thirdly, the non-Lipschitz chaos can be a part of a quantum-inspired computing device. Indeed, it simulates probabilities via physical processes, and it provides influence between different probabilistic branches imitating quantum interference of probabilities (see eqs. (2),(18) and (37)). Moreover, based upon the equivalence between quantum mechanics in imaginary time and classical statistical mechanics in real time one can simulate the Schrödinger equation by the fully deterministic (but unstable) dynamical system (2), (18). The only "non-dynamical" step in such simulations is the. replacement of t by $t\sqrt{-1}$. Hence, formally the dynamical system (2), (18) being considered in a pseudo-eucledian space, can represent a deterministic microstructure behind the Schrödinger equation in the same way in which it represents those for the Fokker-Planck equation in real time,

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